NEURČITÝ INTEGRÁL – SUBSTITUČNÍ METODA

1) $\int \cos 5x\,dx = \left| \begin{array}{l} 5x = t \\ 5dx = dt \\ dx = \frac{dt}{5} \end{array} \right| = \int \cos t \cdot \frac{dt}{5} = \frac{1}{5} \sin t + C$

Substituce zpět: $\frac{1}{5} \sin 5x + C$

2) $\int \sin(8-9x)\,dx = \left| \begin{array}{l} 8-9x = t \\ -9dx = dt \\ dx = \frac{-dt}{9} \end{array} \right| = \int \sin t \cdot \left(\frac{-dt}{9}\right) = +\frac{1}{9} \cos t + C$

Substituce zpět: $\frac{1}{9} \cos(8-9x) + C$

3) $\int 3x \cdot e^{x^2} dx = \left| \begin{array}{l} x^2 = t \\ 2x\,dx = dt \\ x\,dx = \frac{dt}{2} \end{array} \right| = \frac{3}{2} \int e^t dt = \frac{3}{2} e^t + C$

Substituce zpět: $\frac{3}{2} e^{x^2} + C$

4) $\int \operatorname{cotg} x\,dx = \int \frac{\cos x}{\sin x} dx = \left| \begin{array}{l} \sin x = t \\ \cos x\,dx = dt \end{array} \right| = \int \frac{dt}{t} = \ln|t| + C$

Substituce zpět: $\ln|\sin x| + C$

5) $\int \frac{1}{x \cdot \ln^4 x} dx = \int \frac{1}{x} \cdot \frac{1}{\ln^4 x} dx = \left| \begin{array}{l} \ln x = t \\ \frac{1}{x} dx = dt \end{array} \right| = \int \frac{1}{t^4} \cdot dt = \int t^{-4} dt =$

$= \frac{t^{-3}}{-3} + C = \frac{1}{-3t^3} + C$

SZ: $\frac{1}{-3\ln x} + C$

6) $\int \frac{x}{(3+x^2)^2}\,dx = \left| \begin{array}{l} 3+x^2=t \\ 2x\,dx = dt \\ x\,dx = \frac{dt}{2} \end{array} \right| = \int \frac{1}{t^2}\cdot\frac{dt}{2} = \frac{1}{2}\int t^{-2}\,dt = \frac{1}{2}\cdot\frac{t^{-1}}{-1}+C =$

$= \frac{1}{-2t}+C$ sz: $\frac{-1}{2(3+x^2)}+C$

7) $\int \frac{1}{\sqrt{2+3x}}\,dx = \left| \begin{array}{l} \sqrt{2+3x}=t \\ 2+3x=t^2 \\ 3dx = 2t\,dt \\ dx = \frac{2t}{3}dt \end{array} \right| = \int \frac{1}{\cancel{t}}\cdot\frac{2\cancel{t}}{3}\,dt = \frac{2}{3}\int dt = \frac{2}{3}t+C$

Substituce zpět: $\frac{2}{3}\sqrt{2+3x}+C$

8) $\int x^4 e^{2x^5+5}\,dx = \left| \begin{array}{l} 2x^5+5=t \\ 10x^4\,dx = dt \\ x^4\,dx = \frac{dt}{10} \end{array} \right| = \int e^t\cdot\frac{dt}{10} = \frac{1}{10}e^t+C$

sz: $\frac{1}{10}e^{2x^5+5}+C$

9) $\int \frac{1}{x^2}\cdot\cos\frac{1}{x}\,dx = \left| \begin{array}{l} \frac{1}{x}=t \\ -\frac{1}{x^2}dx = dt \end{array} \right| = -\int \cos t\,dt = -\sin t + C$ sz: $-\sin\frac{1}{x}+C$

10) $\int x\cdot\sqrt{x+1}\,dx = \left| \begin{array}{l} \sqrt{x+1}=t \\ x+1=t^2 \\ dx = 2t\,dt \\ x = t^2-1 \end{array} \right| = \int (t^2-1)\cdot t\cdot 2t\,dt = \int (2t^4-2t^2)\,dt =$

$= 2\int t^4\,dt - 2\int t^2\,dt = 2\frac{t^5}{5} - 2\frac{t^3}{3}+C$

Substituce zpět: $\frac{2(\sqrt{x+1})^5}{5} - \frac{2(\sqrt{x+1})^3}{3}+C$

11) $\int 4x^3 \sin x^4 \, dx = \left| \begin{array}{l} x^4 = t \\ 4x^3 dx = dt \end{array} \right| = \int \sin t \cdot dt = -\cos t + C$

Substituce zpět $-\cos x^4 + C$

12) $\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} dx = \left| \begin{array}{l} 2x = t \\ 2dx = dt \\ dx = \frac{dt}{2} \end{array} \right| = \int \frac{1-\cos t}{2} \cdot \frac{dt}{2} =$

$= \frac{1}{4} \int (1-\cos t) \, dt = \frac{1}{4}(t - \sin t) + C$

Tento příklad lze řešit i metodou per partes - viz př. 14.

Sz: $\frac{2x}{4} - \frac{\sin 2x}{4} + C = \frac{2x - (2 \cdot \sin x \cdot \cos x)}{4} + C = \frac{x - \sin x \cdot \cos x}{2} + C$

13) $\int (1+e^{2x})^3 \cdot e^{2x} \, dx = \left| \begin{array}{l} 1+e^{2x} = t \\ 2e^{2x} = dt \\ e^{2x} = \frac{dt}{2} \end{array} \right| = \int t^3 \cdot \frac{dt}{2} = \frac{1}{2} \int t^3 dt =$

$= \frac{1}{2} \cdot \frac{t^4}{4} + C = \frac{t^4}{8} + C$

Sz: $\frac{(1+e^{2x})^4}{8} + C$

14) $\int \frac{\sin 2x}{\sqrt{1+\sin^2 x}} dx = \left| \begin{array}{l} \sqrt{1+\sin^2 x} = t \\ 1+\sin^2 x = t^2 \\ 2 \sin x \cdot \cos x \, dx = 2t \, dt \\ \sin 2x \, dx = 2t \, dt \end{array} \right| = \int \frac{1}{t} \cdot 2t \, dt = 2 \int dt = 2t + C$

Sz: $2\sqrt{1+\sin^2 x} + C$

15) $\int \frac{\cos x - \sin x}{\sin x + \cos x} dx = \left| \begin{array}{l} \sin x + \cos x = t \\ (\cos x - \sin x) dx = dt \end{array} \right| = \int \frac{dt}{t} = \ln|t| + C$

Substituce zpět: $\ln|\sin x + \cos x| + C$

16) $\int \frac{\arcsin^3 x - 3x}{\sqrt{1-x^2}}\,dx = \underbrace{\int \frac{(\arcsin x)^3}{\sqrt{1-x^2}}\,dx}_{I_1} - \underbrace{3\int \frac{x}{\sqrt{1-x^2}}\,dx}_{I_2}$

I_1: $\left| \begin{array}{l} \arcsin x = t \\ \frac{1}{\sqrt{1-x^2}}\,dx = dt \end{array} \right| = \int t^3 \cdot dt = \frac{t^4}{4} + C$ sz: $\frac{1}{4}\arcsin^4 x + C$

I_2: $\left| \begin{array}{l} \sqrt{1-x^2} = s \\ 1-x^2 = s^2 \\ -2x\,dx = 2s\,ds \\ x\,dx = \frac{-\cancel{2}s\,ds}{\cancel{2}} \end{array} \right| = 3\int \frac{-s\,ds}{s} = -3\int ds = -3s + C$ sz: $-3\sqrt{1-x^2} + C$

$I_1 + I_2 = \frac{1}{4}\arcsin^4 x - 3\sqrt{1-x^2} + C$

17) $\int \frac{1}{x}\left(3 + 2x^2 \cdot e^{x^2+1}\right) dx =$ tento příklad je zkouškou pro Lesnickou fakultu

$= \underbrace{3\int \frac{1}{x}\,dx}_{I_1} + \underbrace{2\int x \cdot e^{x^2+1}\,dx}_{I_2}$

I_1: $3\ln|x| + C$

I_2: $\left| \begin{array}{l} x^2+1 = t \\ 2x\,dx = dt \end{array} \right| = \int e^t\,dt = e^t + C$ sz: $e^{x^2+1} + C$

$I_1 + I_2 = 3\ln|x| + e^{x^2+1} + C$

18) $\int \ln(1+\sqrt{x})\,dx = \left| \begin{array}{l} 1+\sqrt{x} = t \\ \sqrt{x} = t-1 \\ x = (t-1)^2 \\ dx = 2(t-1)\,dt \end{array} \right| = \int \ln t \cdot 2(t-1)\,dt = \left| \begin{array}{ll} u' = 2(t-1) & v = \ln t \\ u = (t-1)^2 & v' = \frac{1}{t} \end{array} \right| =$

s. pp

$= \underbrace{(t-1)^2 \cdot \ln t}_{A} - \int (t-1)^2 \cdot \frac{1}{t}\,dt = A - \int (t^2 - 2t + 1) \cdot \frac{1}{t}\,dt = A - \int t\,dx + \int 2\,dt - \int \frac{1}{t}\,dt = \ldots$

sz: $x\ln(1+\sqrt{x}) - \frac{(1+\sqrt{x})^2}{2} + 2(1+\sqrt{x}) - \ln|1+\sqrt{x}| + C$

19) $$\int \frac{\arccos^2 x + x}{\sqrt{1-x^2}}\,dx = \underbrace{\int \frac{(\arccos x)^2}{\sqrt{1-x^2}}\,dx}_{I_1} + \underbrace{\int \frac{x}{\sqrt{1-x^2}}\,dx}_{I_2}$$

I_1: $$\left| \begin{array}{l} \arccos x = t \\ -\frac{1}{\sqrt{1-x^2}}\,dx = dt \end{array} \right| = -\int t^2\,dt = -\frac{t^3}{3} + C \quad \text{sz: } -\frac{(\arccos x)^3}{3} + C$$

I_2: $$\left| \begin{array}{l} \sqrt{1-x^2} = t \\ 1-x^2 = t^2 \\ -2x\,dx = 2t\,dt \\ x\,dx = -t\,dt \end{array} \right| = \int \frac{-t\,dt}{t} = -\int dt = -t + C \quad \text{sz: } -\sqrt{1-x^2} + C$$

$$I_1 + I_2 = -\frac{1}{3}\arccos^3 x - \sqrt{1-x^2} + C$$

20) $$\int \frac{dx}{\cos^2 x\,\sqrt{\operatorname{tg} x - 1}} = \left| \begin{array}{l} \sqrt{\operatorname{tg} x - 1} = t \\ \operatorname{tg} x - 1 = t^2 \\ \frac{1}{\cos^2 x}\,dx = 2t\,dt \end{array} \right| = \int \frac{1}{t}\cdot 2t\,dt = \int 2\,dt = 2t + C$$

Substituce zpět: $2\sqrt{\operatorname{tg} x - 1} + C$

21) $\int e^{-x}\,dx = \left| \begin{array}{l} -x = t \\ -dx = dt \\ dx = -dt \end{array} \right| = -\int e^{t}\,dt = -e^{t} + C$ sz: $-e^{-x} + C$

22) $\int x^4 e^{4x^5+7}\,dx = \left| \begin{array}{l} 4x^5+7 = t \\ 20x^4\,dx = dt \\ x^4\,dx = \frac{dt}{20} \end{array} \right| = \int e^{t} \cdot \frac{1}{20}\,dt = \frac{1}{20} e^{t} + C$

Substituce zpět: $\frac{1}{20} e^{4x^5+7} + C$

23) $\int \frac{10^{\cot g x}}{\sin^2 x}\,dx = \left| \begin{array}{l} \cot g\, x = t \\ -\frac{1}{\sin^2 x}\,dx = dt \end{array} \right| = -\int 10^{t} \cdot dt = -\frac{10^{t}}{\ln 10} + C$

Substituce zpět: $\frac{-10^{\cot g x}}{\ln 10} + C$

24) $\int \frac{dx}{x \cdot \ln^7 x} = \left| \begin{array}{l} \ln x = t \\ \frac{1}{x}\,dx = dt \end{array} \right| = \int \frac{1}{t^7}\,dt = \int t^{-7}\,dt = \frac{t^{-6}}{-6} + C$

S.Z.: $\frac{-1}{6(\ln x)^6} + C$

25)

$\int \frac{e^{2x}\,dx}{\sqrt{e^x - 1}} = \left| \begin{array}{l} \sqrt{e^x - 1} = t \\ e^x - 1 = t^2 \\ e^x\,dx = 2t\,dt \end{array} \right| = \int \frac{(t^2+1) \cdot 2\cancel{t}}{\cancel{t}}\,dt = 2\int (t^2+1)\,dt =$

$= 2\left(\frac{t^3}{3} + t\right) + C = 2t\left(\frac{t^2}{3} + 1\right) + C$

sz: $2\sqrt{e^x - 1}\left(\frac{e^x - 1 + 3}{3}\right) + C = 2\sqrt{e^x - 1} \cdot \left(\frac{e^x + 2}{3}\right) + C$

26) $\int \frac{dx}{\sin^2 3x} = \left| \begin{array}{l} 3x = t \\ 3dx = dt \end{array} \right| = \int \frac{1}{\sin^2 t} \cdot \frac{dt}{3} = \frac{1}{3} \cdot (-\operatorname{cotg} t) + C$

Sz: $\frac{-\operatorname{cotg} 3x}{3} + C$

27) $\int \cos \frac{2x-3}{5} dx = \left| \begin{array}{l} \frac{2x-3}{5} = t \\ \frac{2x}{5} - \frac{3}{5} = t \\ \frac{2}{5} dx = dt \\ dx = \frac{5}{2} dt \end{array} \right| = \int \cos t \cdot \frac{5}{2} dt = \frac{5}{2} \sin t + C$

Sz: $\frac{5}{2} \sin \frac{2x-3}{5} + C$

28) $\int x^3 \cdot \sin x^4 dx = \left| \begin{array}{l} x^4 = t \\ 4x^3 dx = dt \end{array} \right| = \int \sin t \cdot \frac{1}{4} dt = \frac{1}{4} (-\cos t) + C$

Sz: $\frac{-\cos x^4}{4} + C$

29) $\int \frac{dx}{1-x} = \left| \begin{array}{l} 1-x = t \\ -dx = dt \end{array} \right| = -\int \frac{dt}{t} = -\ln|t| + C$ Sz: $-\ln|1-x| + C$

30) $\int \frac{e^x dx}{2+e^x} = \left| \begin{array}{l} 2+e^x = t \\ e^x dx = dt \end{array} \right| = \int \frac{dt}{t} = \ln|t| + C \Rightarrow \ln|2+e^x| + C$

31) $\int \frac{x^3}{x^2-1} dx = \left| \begin{array}{l} x^2-1 = t \\ 2x dx = dt \\ x\, dx = \frac{dt}{2} \end{array} \right| = \int \frac{t+1}{t} \cdot \frac{dt}{2} = \frac{1}{2} \int \left(1 + \frac{1}{t}\right) dt =$ $x^3 = x^2 \cdot x$

$= \frac{1}{2} (t + \ln|t|) + C = \frac{x^2-1}{2} + \frac{\ln|x^2-1|}{2} + C$

32) $$\int \frac{x\,dx}{\sqrt{(x^2-1)^3}} = \left| \begin{array}{l} x^2-1=t \\ 2x\,dx = dt \\ x\,dx = \frac{dt}{2} \end{array} \right| = \frac{1}{2}\int \frac{dt}{\sqrt{t^3}} = \frac{1}{2}\int \frac{dt}{t^{\frac{3}{2}}} = \frac{1}{2}\int t^{-\frac{3}{2}}\,dt =$$

$$= \frac{1}{2}\cdot\frac{t^{-\frac{1}{2}}}{-\frac{1}{2}} + C = \frac{1}{2}\cdot\frac{1}{\sqrt{t}}\cdot(-2) + C$$

Substituce zpět $\underline{\frac{-1}{\sqrt{x^2-1}} + C}$

33) $$\int \frac{x^2\,dx}{(1-x^3)^2} = \left| \begin{array}{l} 1-x^3=t \\ -3x^2\,dx = dt \\ x^2\,dx = \frac{-dt}{3} \end{array} \right| = \int \frac{1}{t^2}\cdot\left(-\frac{1}{3}\right)dt = -\frac{1}{3}\int t^{-2}\,dt =$$

$$= -\frac{1}{3}\cdot\frac{t^{-1}}{-1} + C = \frac{1}{3t} + C \qquad \text{Sj: } \underline{\frac{1}{3(1-x^3)} + C}$$

34) $$\int \frac{x^3}{(2+3x^2)^3}\,dx = \left| \begin{array}{l} x^3 = x\cdot x^2 \\ 2+3x^2=t \\ \hookrightarrow x^2 = \frac{t-2}{3} \\ 6x\,dx = dt \\ x\,dx = \frac{1}{6}dt \end{array} \right| = \int \frac{\frac{t-2}{3}}{t^3}\,\frac{1}{6}dt = \frac{1}{6}\int \frac{t-2}{3}\cdot\frac{dt}{t^3} =$$

$$= \frac{1}{6}\int \frac{t-2}{3t^3}\,dt = \frac{1}{18}\int \frac{t-2}{t^3}\,dt = \frac{1}{18}\int\left(t^{-2}-2t^{-3}\right)dt = \frac{1}{18}\cdot\frac{t^{-1}}{-1} - \frac{\cancel{2}}{18}\cdot\frac{t^{-2}}{\cancel{-2}} + C =$$

$$= \frac{-1}{18t} + \frac{1}{18t^2} + C = \underline{\frac{-1}{18(2+3x^2)} + \frac{1}{18(2+3x^2)^2} + C}$$

35) $$\int \frac{x\,dx}{\sqrt{1-x}} = \left| \begin{array}{l} \sqrt{1-x}=t \\ 1-x=t^2 \\ x = 1-t^2 \\ -dx = 2t\,dt \end{array} \right| = \int \frac{1-t^2}{\cancel{t}}\left(-\cancel{2t}\,dt\right) = -2\int(1-t^2)\,dt =$$

$$= -2\left(t-\frac{t^3}{3}\right) + C = \frac{2t^3}{3} - 2t + C = 2t\left(\frac{t^2}{3}-1\right) + C$$

Sj: $$2\sqrt{1-x}\cdot\left(\frac{1-x}{3}-1\right) + C = 2\sqrt{1-x}\cdot\left(\frac{1-x-3}{3}\right) + C = \underline{\frac{2}{3}\sqrt{1-x}\cdot(-x-2) + C}$$

36) $\displaystyle\int \frac{1}{4+x^2}\ \mathrm{d}x = \left| \begin{array}{rcl} x^2 & = & 4t^2 \\ x & = & 2t \\ \mathrm{d}x & = & 2\ \mathrm{d}t \end{array} \right| = \int \frac{1}{4+4t^2} \cdot 2\ \mathrm{d}t = \int \frac{1}{4\cdot(1+t^2)} \cdot 2\ \mathrm{d}t = \frac{1}{2} \cdot \int \frac{1}{1+t^2}\ \mathrm{d}t = \frac{1}{2} \cdot \operatorname{arctg} t + C =$
$\displaystyle\frac{1}{2} \cdot \operatorname{arctg}\left(\frac{x}{2}\right) + C$

substituce zpět: $\underline{\underline{\displaystyle\frac{1}{2} \cdot \operatorname{arctg}\left(\frac{x}{2}\right) + C}}$

37) $\displaystyle\int \frac{\cos x}{4+\sin^2 x}\ \mathrm{d}x = \left| \begin{array}{rcl} (\sin x)^2 & = & 4t^2 \\ \sin x & = & 2t \\ \cos\ \mathrm{d}x & = & 2\ \mathrm{d}t \end{array} \right| = \int \frac{2\ \mathrm{d}t}{4+4t^2} = \int \frac{2\ \mathrm{d}t}{4\cdot(1+t^2)} = \frac{1}{2} \cdot \int \frac{\mathrm{d}t}{1+t^2} = \frac{1}{2} \cdot \operatorname{arctg} t + C$

substituce zpět: $\underline{\underline{\displaystyle\frac{1}{2} \cdot \operatorname{arctg}\left(\frac{\sin x}{2}\right) + C}}$

38) $\displaystyle\int \frac{\mathrm{e}^{2x}}{\sqrt{\mathrm{e}^x - 1}}\ \mathrm{d}x = \int \frac{\mathrm{e}^x \cdot \mathrm{e}^x}{\sqrt{\mathrm{e}^x - 1}}\ \mathrm{d}x = \left| \begin{array}{rcl} \sqrt{\mathrm{e}^x - 1} & = & t \\ \mathrm{e}^x - 1 & = & t^2 \\ \mathrm{e}^x & = & t^2 + 1 \\ \mathrm{e}^x\ \mathrm{d}x & = & 2t\ \mathrm{d}t \end{array} \right| = \int \frac{t^2+1}{t} \cdot 2t\ \mathrm{d}t = 2 \cdot \int (t^2+1)\ \mathrm{d}t = 2 \cdot \left(\frac{t^3}{3} + t\right) + C$
$\displaystyle= 2 \cdot \left(\frac{(\sqrt{\mathrm{e}^x - 1})^3}{3} + \sqrt{\mathrm{e}^x - 1}\right) + C$

substituce zpět: $\displaystyle 2 \cdot \left(\frac{\sqrt{\mathrm{e}^x - 1} \cdot (\mathrm{e}^x - 1)}{3} + \sqrt{\mathrm{e}^x - 1}\right) + C = \underline{\underline{2 \cdot \sqrt{\mathrm{e}^x - 1} \cdot \left(\frac{\mathrm{e}^x - 1}{3} + 1\right) + C}}$

39) $\displaystyle\int \frac{\ln x}{x \cdot \sqrt{1+\ln x}}\ \mathrm{d}x = \left| \begin{array}{rcl} \sqrt{1+\ln x} & = & t \\ 1+\ln x & = & t^2 \\ \frac{1}{x}\ \mathrm{d}x & = & 2t\ \mathrm{d}t \end{array} \right| = \int \frac{t^2-1}{t} \cdot 2t\ \mathrm{d}t = 2 \cdot \int (t^2-1)\ \mathrm{d}t = 2 \cdot \left(\frac{t^3}{3} - t\right) + C$

substituce zpět: $\displaystyle 2 \cdot \left(\frac{(1+\ln x) \cdot \sqrt{1+\ln x}}{3} - \sqrt{1+\ln^x}\right) + C = 2 \cdot \sqrt{1+\ln x} \cdot \left(\frac{1+\ln x}{3} - 1\right) + C =$
$\displaystyle\underline{\underline{2 \cdot \sqrt{1+\ln x} \cdot \left(\frac{\ln x - 2}{3}\right) + C}}$